On Forward Recursive Estimation for Bivariate Markov Chains

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***Abstract*—A bivariate Markov chain comprises a pair of finite-alphabet continuous-time random processes, which are jointly, but not necessarily individually, Markov. Forward recursive conditional mean estimators are developed for the state, the number of jumps from one state to another, and the total sojourn time of the process in each state. The recursions are implemented using Clark's transformation and tested in estimating the parameter of the bivariate Markov chain using the expectation-maximization (EM) algorithm.** [1]

***Index Terms*—Markov chain, recursive estimation, Zakai equation**

I. Introduction

A bivariate Markov chain comprises a pair of random processes which are jointly Markov. Each of the two processes alone need not be Markov. Each process is assumed continuous-time with finite-alphabet. One of the two processes is assumed observable, possibly through a noisy channel, while the other plays the role of an underlying process. Simultaneous jumps of the two processes are permissible. Examples of finite-state bivariate Markov chains include the Markov modulated Poisson process (MMPP), see, e.g., [10], the Markov modulated Markov process (MMMP), see, e.g., [8], and the batch Markovian arrival process (BMAP), see, e.g., [12]. In all of these examples, the underlying processes are Markov chains. Furthermore, in the MMPP and the MMMP, the observable and underlying processes do not jump simultaneously.

Bivariate Markov chains differ from univariate Markov chains primarily in the distribution of the dwell time of the observable process in each of its states. This distribution was shown to be phase-type in [13]. The set of phase-type distributions is dense in the set of distributions of non-negative random variables [14]. Multivariate Markov chains have been used in modeling ion channel currents, see, e.g., [3], and in modeling DNA sequences in molecular phylogenetics, see, e.g., [4].

In this paper we are interested in forward recursive estimation of certain statistics of the bivariate Markov chain such as the number of jumps from one state to another and the total sojourn time of the chain in each state. These recursions could be embedded in the expectation-maximization (EM) algorithm for maximum likelihood (ML) estimation of the parameter of the bivariate Markov chain [6]. Explicit forward recursions for estimating similar statistics of the MMMP were developed in [9]. Recently [13], forward-backward recursions were developed for ML estimation of the parameter of the bivariate Markov chain using the EM approach. Forward recursions are often preferable since they eliminate the need to store a large amount of training data, and the estimators are updated as the data becomes available.

Forward recursions are usually developed by using the transformation of measure approach in conjunction with the generalized Bayes' rule [16, pp. 259-260]. This approach requires a reference probability measure which dominates the true probability measure of the process of interest. Here this process is the bivariate Markov chain. Under the dominating measure, the two processes comprising the bivariate Markov chain are required to be independent, and hence cannot have simultaneous jumps. Since the bivariate Markov chain could have simultaneous jumps, its probability measure cannot be dominated by any such reference measure. The transformation of measure approach was applicable for MMMPs [9], since, by definition, the two processes of an MMMP cannot jump simultaneously.

This difficulty may be circumvented by assuming that the bivariate Markov chain is observed through a noisy channel, and allowing for sufficiently high signal to noise ratio. We adopt this approach in this paper and develop forward recursions for the conditional mean estimates of the state of the bivariate Markov chain, the number of jumps from each state to another, and the total sojourn time of the chain in each state. The price we have to pay for introducing noise to the observable process is that the resulting recursions involve stochastic integrals which can only be evaluated numerically. This is done here by using Clark's transformation [5] which converts the stochastic differential equations into time-varying ordinary differential equations. The latter are solved numerically. We demonstrate the performance of the forward recursive estimators in an EM parameter estimation setup. We compare the results with those obtained using forward-backward recursions in [13].

The plan for the remainder of this paper is as follows. In Section II we define the bivariate Markov chain, and present some preliminary material such as the Girsanov Theorem for this problem. In Section III we present the recursive estimators for the statistics of the bivariate Markov chain. In Section IV we describe the implementation of the forward recursions. In Section V we provide a numerical example. Comments are

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given in Section VI.

II. Preliminaries

Let $(\Omega, \mathcal{A}, \mathcal{G}, P_1)$ be a complete filtered probability space. On this space, let $Z = \{Z_t : Z_t = (X_t, S_t),\ t \geq 0\}$ denote the bivariate Markov chain, and let $W = \{W_t, t \geq 0\}$ denote a standard Brownian motion. In the absence of noise, $X = \{X_t, t \geq 0\}$ is the observable process of the bivariate Markov chain, and $S = \{S_t, t \geq 0\}$ is its underlying process. The bivariate Markov chain Z is assumed separable, homogeneous, and irreducible. With probability one, all sample paths of Z are right-continuous step functions with a finite number of jumps in each finite interval, see [1, Thm 2.1]. Let

$$Y_t = \frac{1}{\alpha} \int_0^t (1, 0) \begin{pmatrix} X_\tau \\ S_\tau \end{pmatrix} d\tau + W_t, \quad t \geq 0 \tag{1}$$

denote the observed noisy signal where the constant α controls the signal to noise ratio. Let $Z^t = \{Z_\tau, 0 \leq \tau \leq t\}$, and define Y^t in a similar manner. Let $\mathcal{Z}_t = \sigma(Z^t)$ and $\mathcal{F}_t = \sigma(Y^t)$ denote, respectively, the smallest σ-field induced by Z^t and Y^t. Both filtrations are assumed to satisfy the usual condition, i.e., each is right-continuous, [11, p. 47]. Let $\mathcal{G}_t = \mathcal{Z}_t \bigvee \mathcal{F}_t$. This is the smallest σ-field which contains $\mathcal{Z}_t \bigcup \mathcal{F}_t$.

Let $\mathbb{S} = \{a_1, a_2, \ldots, a_r\}$ denote the state space of S where the order r is assumed known. Similarly, let $\mathbb{X} = \{b_1, b_2, \ldots, b_d\}$ denote the state space of X with assumed known order d. The state space of Z is given by $\mathbb{Z} = \mathbb{X} \times \mathbb{S}$. To simplify notation, we may refer to a_i as i for $i = 1, \ldots, r$ and to b_l as l for $l = 1, \ldots, d$. Neither X nor S need be Markov. The generator of the bivariate Markov chain is denoted by $H = \{h_{ln}(ij),\ l, n = 1, \ldots d; i, j = 1, \ldots r\}$ where for $(l, i) \neq (n, j)$,

$$h_{ln}(ij) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} P(Z_{t+\epsilon} = (n, j) \mid Z_t = (l, i)). \tag{2}$$

We do not restrict the form of H and hence the two processes S and X may have simultaneous jumps.

By specializing the generator matrix H, several important special bivariate Markov chains can be obtained. To demonstrate this aspect, we order the state pairs $\{(l, i)\}$ lexicographically, and express the generator as a block matrix $H = \{H_{ln},\ l, n = 1, \ldots, d\}$ where $H_{ln} = \{h_{ln}(ij),\ i, j = 1, \ldots, r\}$ is an $r \times r$ matrix. Consider, for example, the MMMP. For MMMPs, the underlying process S is a Markov chain with generator of say Q, and S and the observable process X cannot jump simultaneously. Hence, the generator H for an MMMP has the following structure. The off-diagonal elements of Q form the off-diagonal elements of each H_{ll}, and all $\{H_{ln}, l \neq n\}$ are diagonal matrices. The underlying process S is Markov if and only if $Q = \sum_{n=1}^d H_{ln}$ for all $l = 1, \ldots, d$ [2]. In another example, the MMPP can also be seen as a particular bivariate Markov chain since it is a particular MMMP with $d = r = 2$ when counts are taken modulo-2 [9].

In this paper, we capitalize on the now standard transformation of measure approach, see, e.g., [19], [16, Chap. 7], to derive forward recursions for conditional mean estimates of certain statistics of the bivariate Markov chain. The approach was recently applied to MMMPs [9]. Let $P_0(Z, Y)$ denote a reference probability measure which dominates the probability measure $P_1(Z, Y)$ of the process. Under P_0, $Y = W$, and Z and Y are statistically independent. The Radon-Nikodym derivative is given by the Girsanov theorem [16] as follows:

$$\begin{aligned} \Lambda_t &= \Lambda(Z^t, Y^t) = \frac{dP_1}{dP_0}(Z^t, Y^t) \\ &= \exp\left\{ \int_0^t \frac{1}{\alpha} X_\tau dY_\tau - \frac{1}{2} \int_0^t \frac{1}{\alpha^2} X_\tau^2 d\tau \right\} \\ &= 1 + \int_0^t \Lambda_{\tau-} \frac{1}{\alpha} X_{\tau-} dY_\tau \end{aligned} \tag{3}$$

and we assume that Novikov's condition is satisfied [11, Thm 8.17]. This condition implies that Λ_t is a $\mathcal{G}_t$-martingale under P_0, and that $\int \Lambda_t dP_0 = 1$.

In this paper we develop forward recursions for estimating the state Z_t, the number of jumps $M_{ij}^{ln}(t)$ from state (l, i) to state (n, j) in $[0, t]$, and the total sojourn time $D_i^l(t)$ in state (l, i) in $[0, t]$, of the bivariate Markov chain, from a sample path Y^t of the noisy signal. The state recursion is similar to that developed in [17] and in [18] for the univariate Markov chain. The recursions for estimating $M_{ij}^{ln}(t)$ and $D_i^l(t)$ could be used in an EM approach for estimating the parameter ϕ of the bivariate Markov chain which is given by the off-diagonal elements of its generator H. When Z is observable in $[0, t]$, then the ML estimate of ϕ is given by [1]

$$\hat{h}_{ln}(ij) = \frac{M_{ij}^{ln}(t)}{D_i^l(t)}, \text{ for } (l, i) \neq (n, j). \tag{4}$$

When the bivariate Markov chain Z is partially observable, i.e., when estimation is performed from Y rather than from Z, then estimation of the parameter can be performed iteratively using the EM approach. In each iteration, $M_{ij}^{ln}(t)$ and $D_i^l(t)$ in (4) are replaced by their conditional mean estimates which are calculated using the current parameter estimate. A numerical example for this EM application is given in Section V.

III. Recursive Estimators

In this section we derive recursions for the state Z_t, the number of jumps $M_{ij}^{ln}(t)$, and the total sojourn time of the bivariate Markov chain D_i^l given Y^t. We discuss their numeric implementation using Clark's transformation in Section IV.

A. State Recursion

Define the indicator function for $Z_t = (X_t, S_t)$ as follows:

$$\varphi_{nj}(t) = \begin{cases} 1 & X_t = b_n, S_t = a_j \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Since Z_t is finite-state, conditional mean estimation of Z_t can be obtained from conditional mean estimation of $\varphi_{nj}(t)$. Using [11, Thm 9.15], the semimartingale representation of $\varphi_{nj}(t)$ is given by

$$\varphi_{nj}(t) = \varphi_{nj}(0) + \sum_{\nu\xi} h_{\nu n}(\xi j) \int_0^t \varphi_{\nu\xi}(\tau) d\tau + V_{nj}(t) \tag{6}$$

where $\{V_{nj}(t)\}$ is a $\mathcal{Z}_t$-martingale. Using the generalized Bayes' rule

$$\hat{\varphi}_{nj}(t) = E_1\{\varphi_{nj}(t) \mid \mathcal{F}_t\} = \frac{E_0\{\varphi_{nj}(t)\Lambda_t \mid \mathcal{F}_t\}}{E_0\{\Lambda_t|\mathcal{F}_t\}} := \frac{\pi_t(\varphi_{nj})}{\pi_t(1)}. \tag{7}$$

The estimate of Z_t is obtained from

$$\hat{Z}_t = E_1\left\{\left(\begin{array}{c} X_t \\ S_t \end{array}\right) \mid \mathcal{F}_t\right\} = \sum_{nj}\left(\begin{array}{c} b_n \\ a_j \end{array}\right)\hat{\varphi}_{nj}(t). \tag{8}$$

Proposition 1. *The state recursion is given by*

$$\begin{aligned}\pi_t(\varphi_{nj}) &= \pi_0(\varphi_{nj}) + \frac{b_n}{\alpha}\int_0^t \pi_{\tau-}(\varphi_{nj})dY_\tau \\ &+ \sum_{li} h_{ln}(ij)\int_0^t \pi_\tau(\varphi_{li})d\tau. \end{aligned}\tag{9}$$

Proof: From the product rule for semimartingales [11, p. 220],

$$\begin{aligned}\varphi_{nj}(t)\Lambda_t = \varphi_{nj}(0)\Lambda_0 + \int_0^t \varphi_{nj}(\tau-)d\Lambda_\tau \\ + \int_0^t \Lambda_{\tau-}d\varphi_{nj}(\tau) + [\varphi_{nj},\Lambda](t)\end{aligned}\tag{10}$$

where $[\varphi_{nj},\Lambda](t)$ denotes the quadratic covariation between $\varphi_{nj}(t)$ and Λ_t. Since $\varphi_{nj}(t)$ is of finite variation and Λ_t is continuous P_0-a.s., we have from [11, p. 219] that $[\varphi_{nj},\Lambda](t)=0$. Substituting (3) in the first integral of (10), and (6) in its second integral, we obtain under P_0

$$\begin{aligned}\varphi_{nj}(t)\Lambda_t &= \varphi_{nj}(0)\Lambda_0 + \frac{1}{\alpha}\int_0^t \varphi_{nj}(\tau-)X_{\tau-}\Lambda_{\tau-}dY_\tau \\ &+\sum_{li} h_{ln}(ij)\int_0^t \Lambda_{\tau-}\ \varphi_{li}(\tau-)d\tau + \int_0^t \Lambda_{\tau-}\ dV_{nj}(\tau). \end{aligned}\tag{11}$$

The conditional mean of (11) under P_0 is obtained as follows. First, consider the last integral in (11). Note that V_{nj} is a $\mathcal{Z}_t$-martingale, and hence also a $\mathcal{G}_t$-martingale, since Z and Y are independent under P_0. Thus, this integral is a $\mathcal{G}_t$-martingale and its conditional mean given $\mathcal{F}_t$ under P_0 equals zero. Next, using Fubini's theorem, and independence of Z and Y under P_0, the conditional mean of each of the remaining integrals in (11) can be applied directly to the integrands and conditioning can be reduced from $\mathcal{F}_t$ to $\mathcal{F}_{\tau-}$. In addition, $\varphi_{nj}(\tau-)X_{\tau-} = \varphi_{nj}(\tau-)b_n$. These observations provide the desired recursion in (9). ∎

When $r=1$ the state estimation problem reduces to that of estimating a univariate Markov chain observed in white Gaussian noise, see [17], [18]. The approach used here adheres to that of [18, Eq. 8].

B. Number of Jumps Recursion

The recursion for estimating the number of jumps $M_{ij}^{ln}(t)$ from state (l,i) to state (n,j) in $[0,t]$ is, in general, an infinite-dimensional estimation problem. It can be turned into a finite-dimensional estimation problem when

$$\eta_{\gamma k}(t) = M_{ij}^{ln}(t)\varphi_{\gamma k}(t), \qquad \begin{array}{l}\gamma = 1,\ldots,d \\ k=1,\ldots,r\end{array} \tag{12}$$

is first estimated, and then summed up over all $\{\gamma,k\}$ to provide the estimate of $M_{ij}^{ln}(t)$ [20], [7]. Let $\pi_t(\eta_{\gamma k}) = E_0\{M_{ij}^{ln}(t)\varphi_{\gamma k}(t)\Lambda_t \mid \mathcal{F}_t\}$. We have,

$$\begin{aligned}\hat{M}_{ij}^{ln}(t) &= E_1\{M_{ij}^{ln}(t) \mid \mathcal{F}_t\} = \frac{E_0\{M_{ij}^{ln}(t)\Lambda_t \mid \mathcal{F}_t\}}{E_0\{\Lambda_t \mid \mathcal{F}_t\}} \\ &= \frac{\sum_{\gamma k}\pi_t(\eta_{\gamma k})}{\pi_t(1)}.\end{aligned}\tag{13}$$

The normalizing factor $\pi_t(1)$ is common to all recursions developed here, and can be obtained from the recursion for $\pi_t(\varphi_{nj})$ in Section III-A.

Proposition 2.

$$\begin{aligned}\pi_t(\eta_{\gamma k}) &= \pi_0(\eta_{\gamma k}) + \frac{b_\gamma}{\alpha}\int_0^t \pi_{\tau-}(\eta_{\gamma k})dY_\tau \\ &+ \sum_{\nu\xi} h_{\nu\gamma}(\xi k)\int_0^t \pi_\tau(\eta_{\nu\xi})d\tau \\ &+ \delta_{jk}\delta_{\gamma n}h_{ln}(ij)\int_0^t \pi_\tau(\varphi_{li})d\tau.\end{aligned}\tag{14}$$

Proof: The proof is developed along the same lines as the proof of the state recursion in Section III-A. It is similar to the proof given in [9] for the number of jumps of the underlying process of an MMMP. The recursion for $\eta_{\gamma k}(t)$ is derived by applying the product rule twice, once for evaluating $M_{ij}^{ln}(t)\Lambda_t$, and then for evaluating $(M_{ij}^{ln}(t)\Lambda_t)\varphi_{\gamma k}(t)$. For the first application we have

$$\begin{aligned}M_{ij}^{ln}(t)\Lambda_t &= M_{ij}^{ln}(0)\Lambda_0 + \int_0^t M_{ij}^{ln}(\tau-)d\Lambda_\tau \\ &+ \int_0^t \Lambda_{\tau-}dM_{ij}^{ln}(\tau) + [M_{ij}^{ln},\Lambda](t),\end{aligned}\tag{15}$$

and we note, using an argument similar to that invoked in (10), that $[M_{ij}^{ln},\Lambda](t)=0$. The differential of Λ_τ is given in (3). The semimartingale representation of $M_{ij}(t)$ is obtained as in [9, Eq. 24] and is given by

$$M_{ij}^{ln}(t) = h_{ln}(ij)\int_0^t \varphi_{li}(\tau-)d\tau + \int_0^t \varphi_{li}(\tau-)dV_{nj}(\tau). \tag{16}$$

On substituting (3) in the first integral of (15), and (16) in its second integral, we obtain

$$\begin{aligned}M_{ij}^{ln}(t)\Lambda_t &= M_{ij}^{ln}(0)\Lambda_0 + \frac{1}{\alpha}\int_0^t M_{ij}^{ln}(\tau-)\Lambda_{\tau-}X_{\tau-}dY_\tau \\ &+ h_{ln}(ij)\int_0^t \Lambda_{\tau-}\ \varphi_{li}(\tau-)d\tau \\ &+ \int_0^t \Lambda_{\tau-}\varphi_{li}(\tau-)\ dV_{nj}(\tau).\end{aligned}\tag{17}$$

Next, the product rule is similarly applied to $(M_{ij}^{ln}(t)\Lambda_t)\varphi_{\gamma k}(t)$. Here, however, $[M_{ij}^{ln}\Lambda,\varphi_{\gamma k}](t)$ differs from zero, and it can be derived similarly to [9, Eq. 27]. We have

$$
\begin{aligned}
&[M^{ln}_{ij}\Lambda, \varphi_{\gamma k}](t) \\
&\quad = (\delta_{n\gamma}\delta_{jk} - \delta_{l\gamma}\delta_{ik}) h_{ln}(ij) \int_0^t \Lambda_{\tau-}\varphi_{li}(\tau-)d\tau \\
&\quad +(\delta_{n\gamma}\delta_{jk} - \delta_{l\gamma}\delta_{ik}) \int_0^t \Lambda_{\tau-}\varphi_{li}(\tau-)dV_{nj}(\tau) \quad (18)
\end{aligned}
$$

where δ_{jk} denotes the Kronecker delta function. Applying the conditional mean $E_0\{\cdot \mid \mathcal{F}_t\}$ to the expression of $(M^{ln}_{ij}(t)\Lambda_t)\varphi_{\gamma k}(t)$, and taking into account that the conditional mean of the martingale integrals involved equal zero, we arrive at (14). ■

C. Total Sojourn Time Recursion

The total sojourn time of the bivariate Markov chain in state (l, i) is given by

$$D^l_i(t) = \int_0^t \varphi_{li}(\tau)d\tau. \quad (19)$$

The estimator of $D^l_i(t)$ is given by

$$\hat{D}^l_i(t) = E_1\{D^l_i(t) \mid \mathcal{F}_t\} = \frac{E_0\{D^l_i(t)\Lambda_t \mid \mathcal{F}_t\}}{E_0\{\Lambda_t | \mathcal{F}_t\}} := \frac{\pi_t(D^l_i)}{\pi_t(1)}. \quad (20)$$

The recursion for $\pi_t(D^l_i)$ is obtained from a recursion for $\zeta_{\gamma k}(t) = D^l_i(t)\varphi_{\gamma k}(t)$ following a similar idea to that used in (12) [20], [7]. We have the following proposition.

Proposition 3.

$$
\begin{aligned}
\pi_t(\zeta_{\gamma k}) &= \pi_0(\zeta_{\gamma k}) + \frac{b_\gamma}{\alpha}\int_0^t \pi_{\tau-}(\zeta_{\gamma k})dY_\tau \\
&+ \sum_{\nu\xi} h_{\nu\gamma}(\xi k)\int_0^t \pi_\tau(\zeta_{\nu\xi})d\tau + \delta_{ik}\delta_{\gamma l}\int_0^t \pi_\tau(\varphi_{li})d\tau. \quad (21)
\end{aligned}
$$

Proof: The product rule is first applied to $D^l_i(t)\Lambda_t$ and then to $(D^l_i(t)\Lambda_t)\varphi_{\gamma k}(t)$. The first application gives

$$
\begin{aligned}
D^l_i(t)\Lambda_t = D^l_i(0)\Lambda_0 &+ \int_0^t D^l_i(\tau-)d\Lambda_\tau \\
&+ \int_0^t \Lambda_{\tau-}dD^l_i(\tau) + [D^l_i, \Lambda](t). \quad (22)
\end{aligned}
$$

Since Λ_t is continuous P_0-a.s., and $D^l_i(t)$ is of finite variation P_0-a.s., $[D^l_i, \Lambda](t) = 0$ P_0-a.s. [11, Thm 1.11]. An expression for $D^l_i(t)\Lambda_t$, is obtained by applying (3) to the first integral of (22), and (19) to its second integral. The resulting expression is then used, along with (6), in the product rule expansion of $D^l_i(t)\Lambda_t\varphi_{\gamma k}(t)$. Here, $[D^l_i\Lambda, \varphi_{\gamma k}](t) = 0$ P_0-a.s. since $\varphi_{\gamma k}$ is of finite variation and $D^l_i(t)\Lambda(t)$ is continuous P_0-a.s. Applying the conditional mean $E_0\{\cdot \mid \mathcal{F}_t\}$, and using the fact that the conditional mean of the martingale involved equals zero, give the desired result. ■

IV. Implementation

Implementation of the recursions in this paper is done using Clark's transformation [5] and numerical integration.

A. State Estimator

Consider implementation of the state recursion (9). Let $\pi_t(\boldsymbol{\varphi})$ be the row vector obtained from concatenation of the row vectors $\{\pi_t(\varphi_{l1}), \ldots, \pi_t(\varphi_{lr})\}$ for $l = 1, \ldots, d$. Define the block diagonal matrix

$$B = \frac{1}{\alpha}\mathrm{diag}[b_1 I_r, \ldots, b_d I_r] \quad (23)$$

where I_r is an $r\times r$ identity matrix. Then, (9) can be vectorized as follows:

$$\pi_t(\boldsymbol{\varphi}) = \pi_0(\boldsymbol{\varphi}) + \int_0^t \pi_{\tau-}(\boldsymbol{\varphi})BdY_\tau + \int_0^t \pi_\tau(\boldsymbol{\varphi})Hd\tau. \quad (24)$$

Clark's transformation converts this stochastic state recursion into a time-varying ordinary differential equation. The transformation, and its differential form obtained by applying Itô's formula [11, p. 111], are given by

$$
\begin{aligned}
L_t &:= \exp\left\{BY_t - \frac{1}{2}B^2 t\right\} \\
&= L_0 + \int_0^t BL_{\tau-}dY_\tau. \quad (25)
\end{aligned}
$$

Now, define $q_t(\boldsymbol{\varphi}) = c_t(\boldsymbol{\varphi})L_t$ where the row vector $c_t(\boldsymbol{\varphi})$ satisfies

$$dc_t(\boldsymbol{\varphi}) = c_t(\boldsymbol{\varphi})L_t H L_t^{-1}dt. \quad (26)$$

From the product rule [11, p. 220]

$$
\begin{aligned}
q_t(\boldsymbol{\varphi}) &= q_0(\boldsymbol{\varphi}) + \int_0^t c_\tau(\boldsymbol{\varphi})dL_{\tau-} \\
&\quad + \int_0^t dc_\tau(\boldsymbol{\varphi})L_{\tau-} + [c(\boldsymbol{\varphi}), L](t) \\
&= q_0(\boldsymbol{\varphi}) + \int_0^t c_\tau(\boldsymbol{\varphi})BL_{\tau-}dY_\tau \\
&\quad + \int_0^t c_\tau(\boldsymbol{\varphi})L_\tau H L_\tau^{-1}L_{\tau-}d\tau + \mathbf{0} \quad (27)
\end{aligned}
$$

where $\mathbf{0}$ denotes a vector of all zeros. Since B and $L_{\tau-}$ are diagonal matrices, $c_\tau(\boldsymbol{\varphi})BL_{\tau-} = q_{\tau-}(\boldsymbol{\varphi})B$. Hence,

$$q_t(\boldsymbol{\varphi}) = q_0(\boldsymbol{\varphi}) + \int_0^t q_{\tau-}(\boldsymbol{\varphi})BdY_\tau + \int_0^t q_\tau(\boldsymbol{\varphi})Hd\tau. \quad (28)$$

Since this equation coincides with (24) when $q_0(\boldsymbol{\varphi}) = \pi_0(\boldsymbol{\varphi})$, we have that $\pi_t(\boldsymbol{\varphi}) = c_t(\boldsymbol{\varphi})L_t$, where $c_t(\boldsymbol{\varphi})$ follows from the solution of the deterministic time-varying differential equation (26). The initial condition of (26) is obtained from $\pi_0(\boldsymbol{\varphi}) = q_0(\boldsymbol{\varphi}) = c_0(\boldsymbol{\varphi})e^{BY_0}$. Hence, $c_0(\boldsymbol{\varphi}) = \pi_0(\boldsymbol{\varphi})e^{-BY_0}$.

A first-order Euler approximation to (26) using $t = k\delta$, where δ is a step size and $k = 0, 1, \ldots$, is given by [5]

$$
\begin{aligned}
c_{k+1}(\boldsymbol{\varphi}) &= c_k(\boldsymbol{\varphi}) + \delta c_k(\boldsymbol{\varphi})L_k H L_k^{-1} \\
q_{k+1}(\boldsymbol{\varphi}) &= c_{k+1}(\boldsymbol{\varphi})L_{k+1}, \quad k = 0, 1, 2, \ldots, \quad (29)
\end{aligned}
$$

where L_k represents $L_{k\delta}$ as obtained from (25). Define $\Delta y_{k+1} = y_{(k+1)\delta} - y_{k\delta}$ where $y_{k\delta}$ represents a realized value of $Y_{k\delta}$. We have from (29) and (25)

$$\begin{aligned} q_{k+1}(\boldsymbol{\varphi}) &= q_k(\boldsymbol{\varphi})(I+\delta H)\exp\left\{B\Delta y_{k+1} - \frac{1}{2}B^2\delta\right\} \\ q_0(\boldsymbol{\varphi}) &= \pi_0(\boldsymbol{\varphi}) \end{aligned} \tag{30}$$

which provide a forward recursion for the unnormalized conditional mean estimate $\pi_k(\boldsymbol{\varphi})$ of the state vector. A recursion for the normalized conditional mean estimate, say $\tilde{\pi}_k(\boldsymbol{\varphi})$, is given by

$$\begin{aligned} \tilde{\pi}_{k+1}(\boldsymbol{\varphi}) &= \frac{\tilde{\pi}_k(\boldsymbol{\varphi})(I+\delta H)\exp\left\{B\Delta y_{k+1} - \frac{1}{2}B^2\delta\right\}}{\tilde{\pi}_k(\boldsymbol{\varphi})(I+\delta H)\exp\left\{B\Delta y_{k+1} - \frac{1}{2}B^2\delta\right\}\mathbf{1}} \\ \tilde{\pi}_0(\boldsymbol{\varphi}) &= \frac{\pi_0(\boldsymbol{\varphi})}{\pi_0(\boldsymbol{\varphi})\mathbf{1}}, \qquad k=0,1,2,\ldots, \end{aligned} \tag{31}$$

where $\mathbf{1}$ denotes a vector of all ones. Normalization is equivalent to the scaling procedure used in [9] to insure numerical stability.

B. Number of Jumps Estimator

Consider implementation of the recursion for the number of jumps given by (14). Let $\pi_t(\boldsymbol{\eta})$ be the row vector obtained from concatenation of the row vectors $\{\pi_t(\eta_{l1}),\ldots,\pi_t(\eta_{lr})\}$ for $l=1,\ldots,d$. Let $\mathbf{1}_{nj}$ denote a unit row vector of size dr with a one in element $(n-1)d+j$ and zeros elsewhere. Then, (14) can be vectorized as follows:

$$\begin{aligned} \pi_t(\boldsymbol{\eta}) &= \pi_0(\boldsymbol{\eta}) + \int_0^t \pi_{\tau-}(\boldsymbol{\eta})B\,dY_\tau + \int_0^t \pi_\tau(\boldsymbol{\eta})H\,d\tau \\ &+ \int_0^t \pi_\tau(\varphi_{li})h_{ln}(ij)\mathbf{1}_{nj}\,d\tau, \end{aligned} \tag{32}$$

where $\pi_\tau(\varphi_{li})$ is the (l,i)th component of $\pi_\tau(\boldsymbol{\varphi})$. Application of Clark's transformation is done similarly to that in Section IV-A, and it results in

$$\begin{aligned} q_{k+1}(\boldsymbol{\eta}) &= [q_k(\boldsymbol{\eta})(I+\delta H) \\ &\quad +\delta q_k(\varphi_{li})h_{ln}(ij)\mathbf{1}_{nj})]\exp\left\{B\Delta y_{k+1} - \frac{1}{2}B^2\delta\right\} \\ q_0(\boldsymbol{\eta}) &= \pi_0(\boldsymbol{\eta}) \end{aligned} \tag{33}$$

which provides a forward recursion for the unnormalized conditional mean estimate $\pi_k(\boldsymbol{\eta})$ of the number of jumps vector. A recursion for the scaled conditional mean estimate can be obtained as in Section IV-A.

C. Total Sojourn Time Estimator

The recursion (21) for the total sojourn time is similar to the recursion (14) for the number of jumps. Hence, its implementation follows closely that of (14) in Section IV-B. Let $\pi_t(\boldsymbol{D})$ be the row vector obtained from concatenation of the row vectors $\{\pi_t(D_1^l),\ldots,\pi_t(D_r^l)\}$ for $l=1,\ldots,d$.

The vector version of (21) is given by

$$\begin{aligned} \pi_t(\boldsymbol{D}) &= \pi_0(\boldsymbol{D}) + \int_0^t \pi_{\tau-}(\boldsymbol{D})B\,dY_\tau \\ &+ \int_0^t \pi_\tau(\boldsymbol{D})H\,d\tau + \int_0^t \pi_\tau(\varphi_{li})\mathbf{1}_{li}\,d\tau. \end{aligned} \tag{34}$$

	ϕ^0		ϕ_0		$\hat{\phi}_{\text{fb}}$	
H_{11}	-70	10	-120	30	-77.03	14.76
	20	-55	2	-8	8.46	-47.95
H_{12}	50	10	70	20	51.80	10.46
	25	10	5	1	32.66	6.83
H_{21}	50	0	70	0	49.50	0
	0	10	0	1	0	9.54
H_{22}	-60	10	-100	30	-59.51	10.01
	20	-30	2	-3	19.61	-29.15

TABLE I
ϕ^0 = true; ϕ_0 = initial; $\hat{\phi}_{\text{fb}}$ = forward-backward recursions.

Application of Clark's transformation is done similarly to that in Section IV-A, and it results in

$$\begin{aligned} q_{k+1}(\boldsymbol{D}) &= [q_k(\boldsymbol{D})(I+\delta H) + \delta q_k(\varphi_{li})\mathbf{1}_{li})] \\ &\quad \cdot\exp\left\{B\Delta y_{k+1} - \frac{1}{2}B^2\delta\right\} \\ q_0(\boldsymbol{D}) &= \pi_0(\boldsymbol{D}), \qquad k=0,1,2,\ldots, \end{aligned} \tag{35}$$

which provides a forward recursion for the unnormalized conditional mean estimate $\pi_k(\boldsymbol{D})$ of the total sojourn time vector. A recursion for the scaled conditional mean estimate can be obtained as in Section IV-A.

V. Numerical Examples

The forward recursions for the bivariate Markov chain were implemented in Python using the SciPy and NumPy libraries. The estimates of $M_{ij}^{ln}(t)$ and $D_i^l(t)$ were used in (4) for EM estimation of the parameter of the bivariate Markov chain. The parameter estimate obtained in this way was compared with the parameter estimate from [13] where $M_{ij}^{ln}(t)$ and $D_i^l(t)$ were estimated using forward-backward recursions. We emphasize that neither approach can be classified as a recursive parameter estimation approach, since each uses the entire data in each EM iteration.

We present numerical results for the example studied in [13, Table 1] and given here in Table I. For this example, $r=d=2$. The generator matrix H is displayed in terms of its block matrix components $\{H_{ln}\}$. The columns labeled ϕ^0, ϕ_0, and $\hat{\phi}_{\text{fb}}$ show, respectively, the true parameter value for the bivariate Markov chain, the initial parameter estimate, and the forward-backward EM estimate from [13].

The bivariate Markov chain parameterized by ϕ^0 in Table I is neither a BMAP nor an MMMP. Indeed, H is not block circulant as in a BMAP, and H_{12} is not diagonal as in an MMMP. Moreover, according to [2, Thm 3.1], the underlying process S is not a homogeneous continuous-time Markov chain since $H_{11}+H_{12}\neq H_{21}+H_{22}$.

The true parameter ϕ^0 was used to generate a bivariate Markov chain sequence with 10^4 jumps in X. In this example, the state space of the process X was given by $b_1=-1$ and $b_2=1$. The observed noisy signal Y was generated according to (1). The realization of Y was then sampled according to a step size δ to obtain an observed noisy sequence $\{y_{k\delta}\}$, which was then applied directly in the forward recursions of Section IV.

	$\hat{\phi}_{\text{for}}$					
δ	7.5×10^{-3}		5×10^{-3}		2.5×10^{-3}	
$\tilde{N}$	$32,102$		$48,153$		$96,305$	
$\tilde{M}$	12		13		13	
H_{11}	-51.07	15.19	-60.31	15.89	-67.03	15.17
	7.22	-43.77	7.45	-43.29	7.76	-45.78
H_{12}	25.32	10.56	33.30	11.11	40.95	10.91
	28.11	8.47	28.21	7.62	31.21	6.81
H_{21}	30.51	0	35.80	0	41.58	0
	0	12.51	0	11.29	0	10.13
H_{22}	-42.73	12.22	-46.79	10.00	-51.52	9.94
	12.81	-25.32	15.97	-27.26	17.90	-28.03

TABLE II
PARAMETER ESTIMATES OBTAINED USING THE FORWARD RECURSIONS WITH $\alpha = 0.01$ AND DIFFERENT STEP SIZES δ.

	$\hat{\phi}_{\text{for}}$					
α	0.1		0.05		0.005	
$\tilde{M}$	200		200		10	
H_{11}	-54.40	15.83	-54.08	17.27	-61.11	15.98
	8.20	-50.74	7.61	-47.61	7.47	-42.79
H_{12}	29.36	9.20	26.44	10.38	34.07	11.05
	30.60	-11.94	32.64	7.36	27.93	7.40
H_{21}	35.18	0	30.50	0	36.33	0
	0	14.41	0	12.63	0	11.06
H_{22}	-49.34	14.16	-37.93	7.44	-47.32	11.00
	10.32	-24.73	11.52	-24.15	15.65	-26.71

TABLE III
PARAMETER ESTIMATES OBTAINED USING THE FORWARD RECURSIONS WITH $\delta = 0.005$ AND DIFFERENT VALUES OF α.

The EM algorithm was terminated either when the relative difference of successive log-likelihood values (see [13]) fell below 10^{-7} or when the number of EM iterations exceeded 200. The estimate $\hat{\phi}_{\text{fb}}$ was obtained after 63 EM iterations.

The parameter estimates obtained using the forward recursive estimators, denoted by $\hat{\phi}_{\text{for}}$, are presented in Table II for $\alpha = 0.01$ and three different step sizes δ. The number of samples contained in the observed noisy sequence $\{y_{k\delta}\}$ is denoted by $\tilde{N}$, and the EM was terminated using the same criterion as in Table I. The number of EM iterations required to obtain each estimate is denoted by $\tilde{M}$. For all three step sizes, the estimates $\hat{\phi}_{\text{for}}$ obtained by the forward recursions are reasonably close to the true parameter ϕ^0 and compare favorably with the estimate $\hat{\phi}_{\text{fb}}$ obtained in [13] using forward-backward recursions. Moreover, the accuracy of the estimates can be seen to improve with decreasing step size δ. At a certain point, however, decreasing the step size further will not improve the accuracy of the estimate, whereas the computational effort required increases linearly with the sampling rate $1/\delta$.

For the estimates shown in Table III, the step size is fixed at $\delta = 0.005$, whereas three different values of α are used. The accuracy of the estimates appears to be much less sensitive to α than to the step size δ. In general, the performance of the forward recursive estimators improves up to a certain point as α decreases. It is interesting to note that the number of EM iterations required was significantly larger for the two larger values of α. On the other hand, if α is too small, numerical overflow will occur in computing samples of the operator L_t in (25).

VI. COMMENTS

The approach taken here for forward recursive estimation of the statistics of the bivariate Markov chain, from a noisy version Y of that process, was motivated by the inherit difficulties in developing such recursions for the observable process X. If simultaneous jumps were not allowed, such as in MMMPs, then the recursions could have had explicit forms, and no numerical integration would have been necessary [9]. The proposed approach appears to perform reasonably well when the step size δ and the gain α are suitably chosen. The computational complexity of these estimators, however, increases linearly with the sampling rate.

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